Math 4200-001
Week 5 concepts and homework
2.2-2.3
Due Friday October 2 at 11:59 p.m.

2.2 : 5, 11.
2.3: 1, 3, 5, 7, 9, 10. In 9b write down a homotopy from the given curve to the standard parameterization of the unit circle, in $\mathbb{C}\backslash\{0\}$, to justify your work.

Math 4200
Friday September 25

2.2 Antiderivatives for analytic functions and Cauchy's Theorem: We'll begin by completing Wednesday's notes on contour algebra and the extension of contour integrals to continuous piece-wise $C^1$ contours $\gamma$. In particular we'll check this extension of the FTC:

<u>Theorem</u> (FTC for contour integrals) Let $A \subseteq \mathbb{C}$ open, $f: A \to \mathbb{C}$ continuous, $\gamma: [a, b] \subseteq \mathbb{R} \to \mathbb{C}$ a piecewise $C^1$ curve. <u>If</u> $f$ has an analytic antiderivative in $A$, i.e. $F' = f$, then complex line integrals only depend on the endpoints of the curve $\gamma$, via the formula

$$\int_\gamma f(z)\,\mathrm{d}z := F(\gamma(b)) - F(\gamma(a))$$

Then the focus of today's notes is to discuss converses to the FTC: namely, what conditions on contour integrals and $f(z)$ imply that $f(z)$ has a complex antiderivative $F(z)$?

Announcements:

Warm-up exercise:

*Contour integrals and antiderivatives:*

Let $f: A \subseteq \mathbb{C} \rightarrow \mathbb{C}$ continuous, $A$ open and connected. When does $f$ have an antiderivative $F(z)$, i.e. $F'(z) = f(z) \; \forall \; z \in A$ ? (Note: we've discussed before why antiderivatives on open connected domains are unique up to additive constants, because their differences have zero derivative.)

<u>Theorem 1</u> The following are equivalent, for $f: A \rightarrow \mathbb{C}$ continuous, where $A$ is open and connected:

(i) $\exists \; F: A \rightarrow \mathbb{C}$ such that $F' = f$ on $A$

(ii) Contour integrals are *path independent*, i.e. for all choices of initial point $P$ and terminal point $Q$ in $A$,

$$\int_{\gamma_0} f(z) \, \mathrm{d}z = \int_{\gamma_1} f(z) \, \mathrm{d}z$$

whenenver $\gamma_0, \gamma_1$ are piecewise $C^1$ (continuous) paths that start at $P$ and end at $Q$.

*proof:*
$(i) \Rightarrow (ii)$ (Use FTC)

$(ii) \Rightarrow (i)$ We are assuming the following:

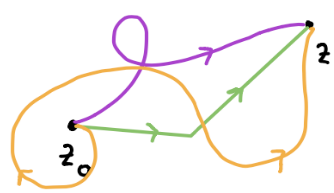
(ii) $f: A \to \mathbb{C}$ continuous, where $A$ is open and connected: Contour integrals for $f$ are *path independent*, i.e. for all choices of initial point $P$ and terminal point $Q$ in $A$,

$$\int_{\gamma_0} f(z)\ \mathrm{d}z = \int_{\gamma_1} f(z)\ \mathrm{d}z$$

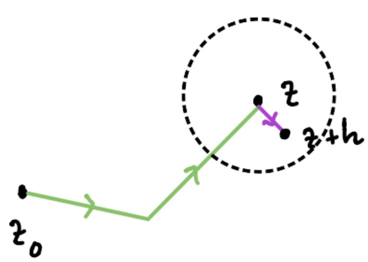
whenenver $\gamma_0, \gamma_1$ are piecewise $C^1$ paths that start at $P$ and end at $Q$.

So, fix any $z_0 \in A$. Because $A$ is open and connected it is *pathwise connected*, and for each $z \in A$ there are piecewise $C^1$ contours in $A$ which start at $z_0$ and end at $z$. (See appendix.) Pick any such contour and denote it by $\gamma_{z_0 z}$. Define our candidate antiderivative by

$$F(z) = \int_{\gamma_{z_0 z}} f(\zeta)\ \mathrm{d}\zeta,$$




By hypothesis $F(z)$ is well-defined, since contour integrals are path-independent. Our work is to show that $F$ is complex differentiable at each $z \in A$ and that its derivative is $f$. We'll verify the affine approximation formula for $F$!



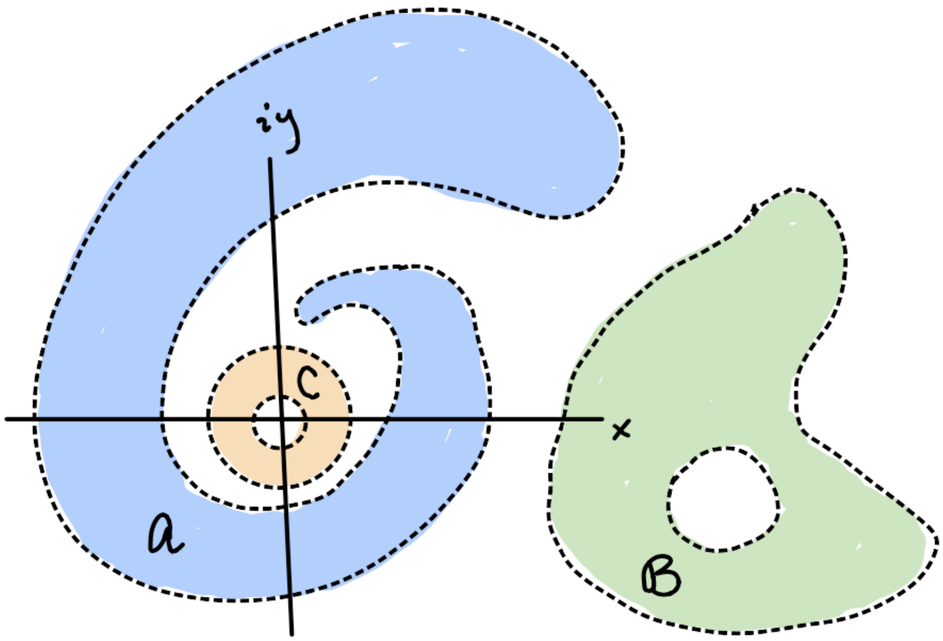
Theorem 2 If $A$ is *open* and *simply connected*. Let $f: A \to \mathbb{C}$ be analytic and $C^1$. Then $f$ has antiderivatives $F$, unique up to additive constants.

*proof:* We'll use Green's Theorem to explain why the path-independence condition (ii) of Theorem 1 holds. Thus antiderivatives exist, and one way to express them is via contour integrals as in the previous discussion:

$$F(z) = \int_{\gamma_{z_0 z}} f(\zeta)\, d\zeta$$

Notice how we will use the "no-holes" idea of *simply-connected*. This explanation is not completely rigorous, but we'll fix that lack of rigor in section 2.3 by defining simply connected more carefully, and also by using different techniques that don't depend on Greens' Theorem and our heuristic pictures of what contours look like.

*Example (like hw due today)* Discuss whether it is possible to define $\log(z)$ as an analytic (single-valued) function on each of the following three domains:



*Appendix:* Connected domains, path connected domains, simply connected domains:
Some Math 3220/Chapter 1.4 analysis background material we need now:

Recall that a domain $A \subseteq \mathbb{C}$ is called *connected* iff there is no disconnection of $A$ into disjoint (relatively) open and non-empty subsets $U, V$ i.e. such that

$$A = U \cup V$$
$$U \cap V = \emptyset.$$

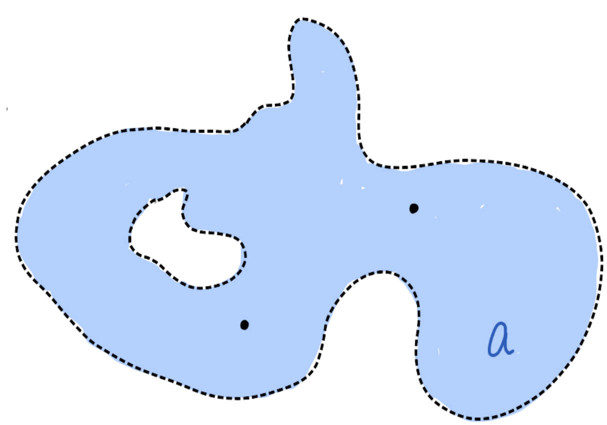
If we restrict to open domains $A$, then subsets $U, V$ that are relatively open are actually open.

There is a related definition:
*Definition* A subset $A \subseteq \mathbb{C}$ is called *path connected* iff $\forall\ P, Q \in A$, there exists a continuous path $\gamma : [a, b] \to A$ such that $\gamma(a) = P,\ \gamma(b) = Q$.

<u>Theorem</u> Let $A \subseteq \mathbb{C}$ be open. Then $A$ is connected if and only if $A$ is path connected. Furthermore, if $A$ is connected then there are piecewise $C^1$ paths connecting all possible pairs of points in $A$. (Analogous theorem holds in $\mathbb{R}^n$.)
*proof:* $\Rightarrow$ : Let $A$ be connected and open. We will show it is path connected, with piecewise $C^1$ paths. Pick any base point $z_0 \in A$. Define $U$ to be the set of points that can be connected to $z_0$ with a piecewise $C^1$ path. $U$ is non-empty since $D(z_0; r) \subseteq U$ as long as $r$ is small enough so that the disk is in $A$. In fact, for all $z \in D(z_0; r)$ we can use the straight-line paths

$$\gamma(t) = z_0 + t(z - z_0), \quad 0 \le t \le 1$$

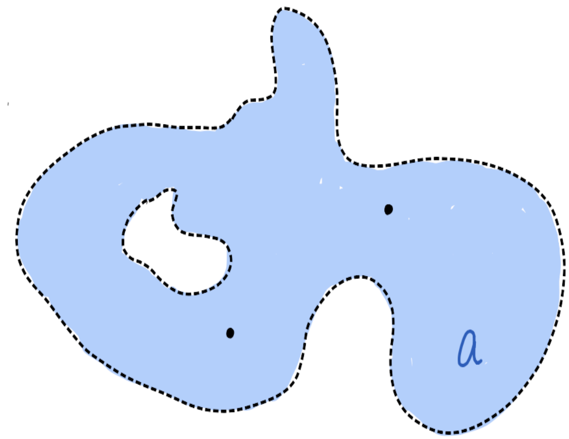
to connect $z_0$ to $z$.



The proof that $U$ is open is analogous: Let $z \in U$ and let $\gamma$ be a piecewise $C^1$ path connecting $z_0$ to $z$. Then for $w \in D(z, r) \subseteq A$ and

$$\gamma_1(t) = z + t(w - z), \quad 0 \le t \le 1$$

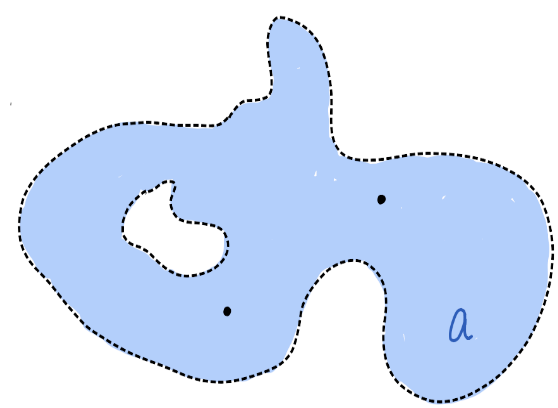
the combined path $\gamma + \gamma_1$ is a piecewise $C^1$ path connecting $z_0$ to $w$. Thus $U$ is open.




But the complement $V := A \setminus U$ is open by a similar argument: If $V$ is non-empty, let $z_1 \in V$, $\mathrm{D}(z_1; r) \subseteq A$. Then $\mathrm{D}(z_1, r) \subseteq V$ as well, since if $\exists\, z \in U \cap D(z_1; r)$ there is a piecewise $C^1$ path $\gamma$ from $z_0$ to $z$, and letting

$$\gamma_2(t) = z + t(z_1 - z), \quad 0 \le t \le 1,$$

the path $\gamma + \gamma_2$ would connect $z_0$ to $z_1$. Thus, since $A$ is connected, we must have that $V = A \setminus U$ is empty.



*path connected implies connected:*
Let $A$ be path connected. Let $A = U \cup V$ with $U$, $V$ open, $U$ non-empty, and $U \cap V = \varnothing$. We will show $V$ is empty. If not, pick $P \in U$, $Q \in V$, and let

$$\gamma : [a, b] \to \mathbb{C}$$

be a continuous path connecting $P$ to $Q$, i.e. $\gamma(a) = P$, $\gamma(b) = Q$. Let $T \in [a, b]$ be defined by

$$T := sup\{t \in [a, b] \mid \gamma([a, t]) \subseteq U\}$$

Because $U$ is open, $T > a$. If $T = b$ we would have $Q \in U$ which would be a contradiction. But if $a < T < b$ then $\gamma(T)$ is in neither $U$ nor $V$: If $\gamma(T) \in U$ then by continuity and $U$ open, there exists $\delta > 0$ so that $\gamma([T, T + \delta) \subseteq U$, hence $\gamma([a, T + \delta) \subseteq U$, contradicting the definition of $T$. Similarly, if $\gamma(T) \in V$, continuity of $\gamma$ and $V$ open implies there exists $\delta > 0$ so that $\gamma([T - \delta, T] \subseteq V$, another contradiction. Thus $T$ can't exist, and $V$ must be empty.